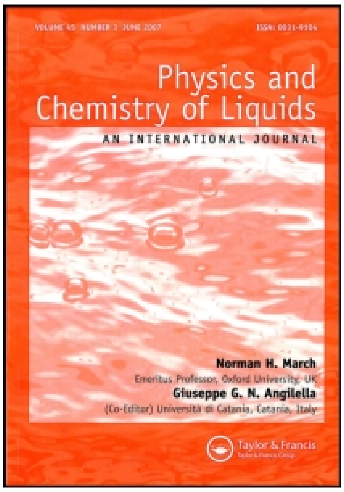

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Some current issues concerning statics and dynamics of inhomogeneous superfluid gases

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Review

SOME CURRENT ISSUES CONCERNING STATICS AND DYNAMICS OF INHOMOGENEOUS SUPERFLUID GASES

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After a brief introduction to the static Gross–Pitaevskii (GP) differential equation for the ground state of a Bose–Einstein condensed gas and an example of its use in describing a vortex array in a trapped condensate, a recent generalization is reviewed. By starting from the Bogoliubov–de Gennes equations for superfluid fermions, an integral equation results. This is shown to contain the static GP equation as a limiting case, by means of a low-order gradient expansion of the order parameter.

The time-dependent GP approximation is then presented and illustrated by a numerical example concerning drop emission under gravity from a gaseous condensate in an optical lattice. Further selected topics of present interest relating more generally to the dynamics of trapped superfluid gases concern the effects of temperature on collective excitations in a Bose gas and the fingerprints that these excitations in a gas of paired fermions contain of the crossover from a Bardeen–Cooper–Schrieffer correlated assembly to a Bose–Einstein condensate of composite bosons. Finally, some directions are pointed in which interaction of first-principles theory with experiment should prove fruitful.

Keywords: Superfluids; Bose–Einstein condensates; Fermi gases; Gross–Pitaevskii equation and beyond

1. BACKGROUND AND OUTLINE

The experimental attainment of Bose–Einstein condensation in ultracold vapours of alkali atoms has resulted in an explosion of activity in the study of dilute atomic gases in condensed quantum form inside magnetic traps and optical lattices. The latest experimental achievements have been the observation of a Bose–Einstein condensate (BEC) in a molecular gas formed from paired fermionic atoms, and the use of tunable fermion–fermion interactions in the exploration of the crossover from a Bardeen–Cooper–Schrieffer (BCS) picture to a BEC picture of a superfluid of composite bosons. A very recent review by Minguzzi *et al.* [1] has given a definitive survey of much of the progress made so far, with special emphasis on numerical

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methods for atomic quantum gases and their applications to Bose–Einstein condensates and to ultracold fermions.

Our aim in the present, much shorter, review is essentially complementary. While common ground with [1] is emphasis on degenerate quantum gases under confinement within an *s*-wave scattering picture for the (repulsive or attractive) atom–atom collisions, we shall at first restrict ourselves to the extremely low temperature and high dilution limit in which a weakly coupled Bose gas is essentially entirely Bose–Einstein condensed. We shall give some attention to the relation between the mean-field Gross–Pitaevskii (GP) equation describing many of the physical properties of such a gas and other many-body formulations, such as the Bogoliubov–de Gennes equations for superfluid fermions [2] and the BCS model [3]. We shall then extend the discussion to selected topics concerning the dynamics of confined quantum gases, first still within the GP frame and then transcending it to deal with bosons at finite temperature and with superfluids of paired fermions. The reader interested in a general overview on quantum gases and more specifically in numerical studies, apart from individual examples of static and dynamic phenomena described by the GP equation in Sections 2 and 4 below, is advised first to refer to the review of Minguzzi *et al.* [1].

In a little more detail, the outline of the present article is then as follows. In Section 2 the customary variational principle leading to the GP equation for the ground state of a Bose gas is set down and interpreted. A numerical solution in a physical situation connected with multi-vortex ordering is referred to briefly, in relation to experimental findings [4]. Section 3 is devoted to a recently proposed integral equation [5], designed to include effects of non-locality in a derivation of the GP equation from the Bogoliubov–de Gennes equations for superfluid fermions. Section 4 extends the discussion from the static GP theory to the case of time dependence by starting from a second-quantized form of the Hamiltonian. In this case, the selected illustrative example of an application of the time-dependent GP equation concerns the emission of drops of coherent atomic matter from a BEC in an optical lattice under the drive of gravity [6]. As important directions of very recent studies transcending the GP framework, collective excitations are then discussed in Section 5. We review here a study of the dynamics of a trapped Bose gas at finite temperature, which treats the fluctuations in the condensate and in the thermal cloud on an equal footing and demonstrates that combined normal and anomalous density fluctuations play a crucial role in the low-lying monopolar excitations [7]. In a further example we discuss how collective excitations in a superfluid from fermion pairing contain information on the crossover from a molecular BEC to a BCS correlated assembly [8]. Section 6 finally offers a summary and also some suggestions for directions in which first-principles theory should lead to fruitful interaction with experiment.

2. VARIATIONAL PRINCIPLE AND THE STATIC GP EQUATION

The time-independent GP equation for a Bose-condensed gas in a three-dimensional trap at zero temperature can be regarded as resulting from variational minimization of the energy functional

$$E[\Phi] = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\Phi|^2 + V_{\mathrm{ext}}(\mathbf{r})|\Phi|^2 + \frac{g}{2}|\Phi|^4 \right]. \tag{1}$$

The 'condensate wave function' $\Phi(\mathbf{r})$ is the order parameter of the system and $|\Phi(\mathbf{r})|^2$ gives the inhomogeneous density profile. On the RHS of Eq. (1), the terms successively represent (i) the kinetic energy of the condensate, (ii) the potential energy due to the external confinement, and finally (iii) the mean-field interaction energy for contact atom–atom interactions with coupling constant $g = 4\pi\hbar^2 a/m$, a being the s-wave scattering length. For repulsive interactions ($g > 0$) the functional is convex and the minimum corresponds to the stable ground state. For the case when $g < 0$, the ground state exists only at weak coupling for a limited number of trapped bosons, as long as the zero-point energy can balance the effect of attractions and prevent collapse. For some details and references of a similar density functional approach for low-dimensional Bose gases, where the interaction energy has a different dependence on density (e.g. $\ln|\Phi|^2$ in the two-dimensional case) the reader should again refer [1].

Minimizing Eq. (1) with respect to the order parameter, one finds the static GP equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\Phi(\mathbf{r})|^2\right]\Phi(\mathbf{r}) = \mu\Phi(\mathbf{r}), \tag{2}$$

a condensate at equilibrium being at the chemical potential μ.

2.1. Numerical Application of Static GP Equation to a Vortex Array

We shall report one application of the static GP equation, which we have chosen as the vortex array obtained by numerical solution of the two-dimensional differential equation for a rotating BEC under tight vertical confinement [4]. One then in this rotating condensate, described in the rotating frame by such a static GP equation, needs to include an inertial term. Thus Eq. (2) is extended as

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\Phi(\mathbf{r})|^2 + \Omega L_z\right]\Phi(\mathbf{r}) = \mu\Phi(\mathbf{r}), \tag{3}$$

where Ω is the rotational frequency and L_z is the angular momentum component along the rotational axis. Numerical solution of Eq. (3) needs special care in handling the term ΩL_z, for as emphasized in the study of Castin and Dum [4] this term is not diagonal in position or in momentum space.

Beginning with different trial states, one can produce single-vortex as well as multi-vortex solutions. In the latter type of solution, the vortices shows a tendency to arrange themselves into a triangular configuration, as depicted in Fig. 1 redrawn from Castin and Dum [4]. As discussed by Minguzzi *et al.* [1], an important feature which has emerged from the solution of the static GP equation in a cigar-shaped trap is that a vortex line can be bent (see Fig. 17 of [1], following Garcia-Ripoll and Pèrez-Garcia [9]). This is in general accord with experimental findings.

3. PROPOSED INTEGRAL EQUATION TRANSCENDING STATIC GP EQUATION FOR PAIRED FERMIONS

A derivation of the static GP equation (2) has recently been given by Pieri and Strinati [10] from the Bogoliubov–de Gennes equations for superfluid fermions. The starting

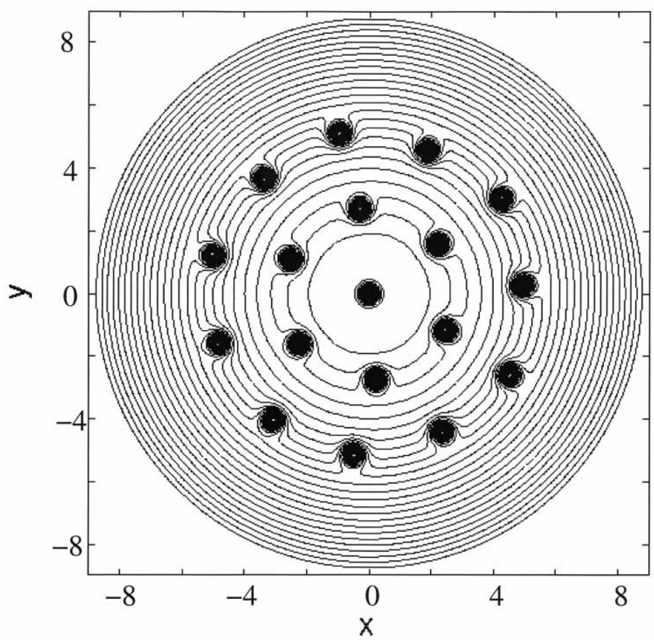

FIGURE 1 Depicts vortex array found by numerical solution of a Gross–Pitaevskii equation for a rotating Bose–Einstein condensate. Redrawn from Castin and Dum [4].

point consists of coupled integral equations involving three Green functions, i.e. the normal and anomalous single-particle Green functions, while the third Green function, denoted by $\tilde{G}_0$, satisfies the equation

$$[i\omega_s - H(\mathbf{r})]\tilde{G}_0(\mathbf{r}, \mathbf{r}'; \omega_s) = \delta(\mathbf{r} - \mathbf{r}'). \tag{4}$$

Here $\omega_s = (2s+1)\pi/\beta$, where s is an integer, denotes a fermionic Matsubara frequency, and β is the reciprocal of the thermal energy $k_B T$. Finally in Eq. (4) the single-particle Hamiltonian $H(\mathbf{r})$ is defined by $H(\mathbf{r}) = -\hbar^2\nabla^2/2m + V_{\text{ext}}(\mathbf{r}) - \mu$, where μ is the fermionic chemical potential. The coupled integral equations involving the above three Green functions, when taken together with the self-consistent equation for the gap function

$$\Delta^*(\mathbf{r}) = \frac{V_0}{\beta}\sum_s G_{21}(\mathbf{r},\mathbf{r}; \omega_s), \tag{5}$$

G_{21} being the anomalous Green function, are entirely equivalent to the Bogoliubov–de Gennes equations. These, of course, describe the behaviour of superfluid fermions in the presence of an external potential. The constant $V_0 < 0$ entering Eq. (5) arises from an attractive contact interaction between fermions with opposite spins.

The ratio of gap function to chemical potential is now used as an expansion parameter, which then leads for strong coupling to an integral equation

$$-\frac{\Delta^*(\mathbf{r})}{V_0} = \int d\mathbf{r}_1\, Q(\mathbf{r}, \mathbf{r}_1)\Delta^*(\mathbf{r}_1) + \int d\mathbf{r}_1\, d\mathbf{r}_2\, d\mathbf{r}_3\, R(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\Delta^*(\mathbf{r}_1)\Delta(\mathbf{r}_2)\Delta^*(\mathbf{r}_3). \tag{6}$$

The four-point function R is written explicitly in terms of $\tilde{G}_0$ in Eq. (15) of the work of Pieri and Strinati [10]. In terms of the same Green function, Q is also given by

$$Q(\mathbf{r}, \mathbf{r}') = \beta^{-1} \sum_s \tilde{G}_0(\mathbf{r}', \mathbf{r}; -\omega_s)\tilde{G}_0(\mathbf{r}', \mathbf{r}; \omega_s). \tag{7}$$

3.1. Beyond Semi-classical Approximation to Green Function $\tilde{G}_0$

In [10] it was shown that, by invoking semi-classical Thomas–Fermi-like approximations to the Green function $\tilde{G}_0$, one can recover from the Bogoliubov–de Gennes equations the time-independent GP equation (2) set out in the preceding section.

Subsequently, Angilella *et al.* [5] have taken the integral equation (6) as their starting point but have calculated the non-local kernel $Q(\mathbf{r}, \mathbf{r}')$ without recourse to semi-classical approximations to $\tilde{G}_0(\mathbf{r}, \mathbf{r}'; \omega_s)$. They then derive an integral equation that implies less restrictive assumptions than those made in [10]. The main step is to perform the sum over Matsubara fermionic frequencies in Eq. (7) by utilizing the early work of Stoddart *et al.* [11], in which a procedure was derived for calculating (essentially the inverse Laplace transform of) the Green function in the presence of an external potential without any Thomas–Fermi-like assumption of slow spatial variations.

The outcome of the study of Angilella *et al.* [5] is then a proposed integral equation for the condensate wave function $\Phi(\mathbf{r})$ in a gas of strongly paired fermions, which is

$$-\frac{\Phi^*(\mathbf{r})}{V_0} = \int d\mathbf{r}_1\, Q(\mathbf{r}, \mathbf{r}_1)\Phi^*(\mathbf{r}_1) + k^{-2} \int d\mathbf{r}_1\, d\mathbf{r}_2\, d\mathbf{r}_3\, R(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\Phi^*(\mathbf{r}_1)\Phi(\mathbf{r}_2)\Phi^*(\mathbf{r}_3), \tag{8}$$

where the essential point is to retain the non-local kernel $Q(\mathbf{r}, \mathbf{r}')$ as a function of the external potential $V_{\text{ext}}(\mathbf{r})$. The assumption has been made that the condensate wave function is related to the gap function by $\Phi(\mathbf{r}) = k\Delta(\mathbf{r})$. As difficulties still centre around the evaluation of the four-point function R entering Eq. (8), Angilella *et al.* retain the approximation given by the Pieri–Strinati development in the 'smallest' term involving $O(\Phi^3)$ in the RHS of their integral equation. Thus one is led to the still non-local equation

$$-\frac{\Phi^*(\mathbf{r})}{V_0} = \int d\mathbf{r}_1\, Q(\mathbf{r}, \mathbf{r}_1)\Phi^*(\mathbf{r}_1) - \frac{ma_F^2}{2}|\Phi(\mathbf{r})|^2\Phi^*(\mathbf{r}), \tag{9}$$

where $a_F \approx (2m|\mu|)^{-1/2}$ represents the length scale for the non-interacting Green function.

3.2. Recovery of GP Differential Equation from Integral Equation Theory

We next outline how one can pass from the integral equation formulation to the static GP equation. Let us, to illustrate the essence of the matter, focus on the term $\int d\mathbf{r}_1 \times Q(\mathbf{r},\mathbf{r}_1)\Phi^*(\mathbf{r}_1)$ appearing in both Eqs. (8) and (9).

Then, one approximates this term containing the non-local kernel $Q(\mathbf{r}, \mathbf{r}_1)$ by assuming that $\Phi^*(\mathbf{r}_1)$ is sufficiently slowly varying spatially to allow a gradient expansion to be made around the point $\mathbf{r}$. The result is that one can write

$$\int d\mathbf{r}_1\, Q(\mathbf{r}, \mathbf{r}_1)\Phi^*(\mathbf{r}_1) \cong [A(\mathbf{r}) + B(\mathbf{r})\nabla^2]\Phi^*(\mathbf{r}). \tag{10}$$

Then it readily follows that $B(\mathbf{r}) \propto \int d\mathbf{r}_1\, Q(\mathbf{r}, \mathbf{r}_1)|\mathbf{r}_1 - \mathbf{r}|^2 \propto m a_F$ while, with a further Thomas–Fermi-like approximation, $A(\mathbf{r})$ is equal to a constant plus a term proportional to the potential $V_{\mathrm{ext}}(\mathbf{r})$. One is led back to precisely the structure of Eq. (2).

The GP formulation and its non-local extension by Angilella *et al.* [5] are valid in the strong coupling limit of fermion superfluidity. It may be recalled that in the weak coupling limit one can derive a Ginzburg–Landau equation starting again from the Bogoliubov–de Gennes equations, an explicit derivation for the case of a harmonic trap having been given by Baranov and Petrov [12]. In this weak-coupling regime, the normal state is a degenerate Fermi liquid that undergoes a pairing instability at a temperature $T_c = T_{\mathrm{BCS}}$ such that $k_{\mathrm{B}} T_c$ is much smaller than the ground-state energy of the non-interacting Fermi gas. In this regime the formation of Cooper pairs and their condensation occur simultaneously at the transition. In the strong coupling limit a BEC of bosons is instead obtained, these being composite objects which are made up of an even number of fermions and condense at a temperature T_c much below their dissociation temperature. Of course, these composite bosons may be weakly coupled and the GP framework then applies in the appropriate situation of high dilution and essentially zero temperature.

Of great topical interest is the crossover between these two regimes of attractive fermion–fermion coupling in trapped gases of fermionic atoms. The article of Minguzzi *et al.* [1] can again be referred to for an introductory review on theoretical models and numerical studies of the critical temperature and of the equilibrium properties of a Fermi gas with attractive interactions in this crossover from a BCS superfluid phase of weakly bound fermions to a BEC of strongly bound composite bosons.

3.3. The Relation between the GP and Bogoliubov Descriptions of a Dilute Bose Gas

To conclude this part of the present review, we draw attention to the quite recent study of Leggett [13] concerning the question as to whether a genuine many-body wave function underlies the static GP method. Leggett formulates what he terms a 'pseudo-paradox' in the theory of a dilute Bose gas with repulsive interactions: that is, the ground-state energy within the GP approximation is lower than that in the Bogoliubov treatment. From a variational viewpoint one might be led to think that the former should afford the better approximation to the true ground state, which Leggett points out is in opposition to the established wisdom concerning this problem.

Leggett proceeds to demonstrate that the pseudo-paradox is resolved by a correct transcription of the two-body scattering theory to the many-body case. He reaches the conclusion that the GP approximation does not correspond to any well-defined Ansatz for the many-body wave function. The latter will have to optimally accommodate the Fock exchange contribution to the total energy, which is instead absent in the two-particle problem.

4. DYNAMICS OF INHOMOGENEOUS BOSE SUPERFLUIDS IN THE TIME-DEPENDENT GP EQUATION

We turn to consider some current issues concerning the dynamics of trapped superfluid boson gases. Again, it is natural to start with the time-dependent GP equation, for which we report a standard derivation together with a specific illustration of its numerical solution.

Having obtained the static GP equation in Section 2 from an energy variational principle, let us approach the time-dependent case in the form of the equation for the condensate wave function $\Phi(\mathbf{r},t)$ by starting from the second-quantized Hamiltonian H. This reads, in conventional notation denoting the external potential by $V_{\text{ext}}(\mathbf{r})$ and with $U(\mathbf{r}-\mathbf{r}')$ as the two-body interatomic potential:

$$H = \int d\mathbf{r}\, \hat{\Psi}^{\dagger}(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r})\right]\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int\int d\mathbf{r}\, d\mathbf{r}'\, \hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}^{\dagger}(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}), \tag{11}$$

where $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^{\dagger}(\mathbf{r})$ are the bosonic field operators. The evolution equation for the operator $\hat{\Psi}^{\dagger}(\mathbf{r},t)$ then follows in the form

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + \int d\mathbf{r}'\, \hat{\Psi}^{\dagger}(\mathbf{r}',t)U(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}',t)\right]\hat{\Psi}(\mathbf{r},t). \tag{12}$$

In the limit of a highly dilute and fully Bose–Einstein condensed gas at zero temperature, the field operator is replaced by the classical field $\Phi(\mathbf{r},t)$ and the interatomic pair potential is taken as a contact interaction from *s*-wave scattering (cf. Section 2). One then is directly led to the time-dependent equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\right]\Phi(\mathbf{r},t), \tag{13}$$

which was derived independently by Gross [14] and Pitaevskii [15]. Writing for the ground state $\Phi(\mathbf{r},t)=\Phi(\mathbf{r})\exp(-i\mu t/\hbar)$ one is led back to the static GP equation. The collective excitations of the condensate are described by time-dependent deviations from this ground state.

It should be noted that the time-dependent GP equation omits dissipation and cannot deal with the dynamics of a confined Bose gas when quantum depletion or thermal excitations are important. Examples of such situations are condensate formation and decay, phase decoherence, damping of collective motions, vortex formation, and excitations from a non-mean-field ground state. The GP framework has nevertheless proved to be useful in dealing with a variety of dynamical phenomena in condensate clouds, such as collective-mode frequencies in harmonic or optical-lattice confinement, interference patterns, transport and tunnelling through an optical lattice, vortex dynamics, soliton propagation, shock-wave dynamics, four-wave mixing, atom-laser output, and expansion of a rotating condensate. By way of illustration of the usefulness of Eq. (13), the next subsection records an example of coherent transport through an array of potential wells representing an optical lattice as revealed by the emission of drops of matter under gravity.

4.1. Drop Emission from an Optical Lattice Under Gravity

The extensive numerical study of Chiofalo *et al.* [6] was motivated by an experiment by Anderson and Kasevich [16], in which a nearly pure BEC was poured from a magneto-optic trap into a vertical optical lattice produced by a detuned standing wave of light from two counter-propagating laser beams. The gravitational field tilts the lattice potential and drives tunnelling from well states to the continuum. Interference occurring between coherent blobs of condensate at different lattice sites results in the appearance of falling drops, which can be interpreted as coherent matter-wave pulses by analogy with a mode-locked photon laser. The analogue of the laser cavity is the Brillouin zone in momentum space, so that the modulation time for the pulsed emission of drops is equal to the period of Bloch oscillations. In the specific conditions of both the experiment and the numerical study the time interval between successive drops is 1.1 ms, which is indeed the period of Bloch oscillations for a quasi-particle (the whole coherent condensate) driven by the constant force of terrestrial gravity through a periodic array of potential wells with a lattice period equal to one-half of the optical laser wavelength λ.

In their numerical study Chiofalo *et al.* [6] pay primary attention to the role of the atomic interactions in this phenomenon of coherent transport. The period of drop emission depends only on the intensity of the drive and on the lattice spacing, in agreement with the band-structure theory of transport in periodic structures. The height of the lattice barrier and the strength of the interatomic forces instead determine the size and shape of the emitted drops. Figure 2, redrawn from [6], shows the density profile of the drops emitted from a non-interacting BEC after 4.6 ms, as a function of a suitably scaled distance z. The main condensate around the origin at $z = 0$ has

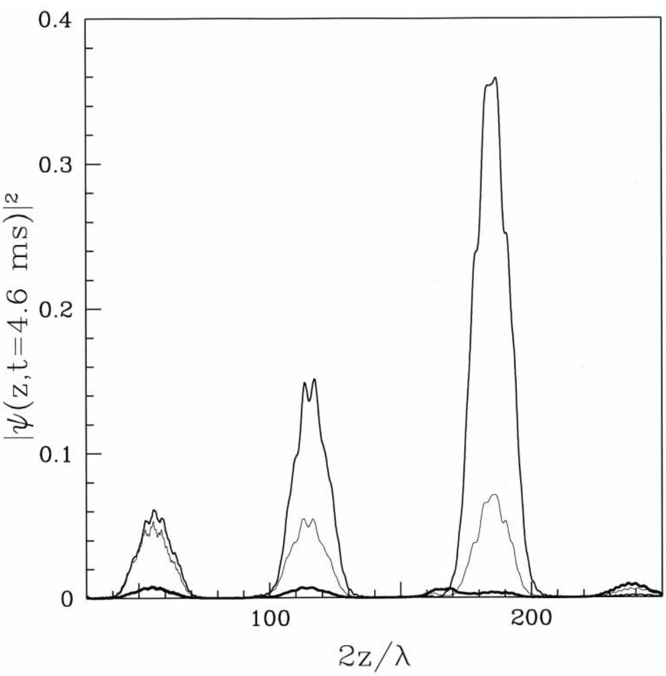

FIGURE 2 Relates to condensate drop emission from an optical lattice under gravity. The figure shows the density profile of the condensate drops after 4.6 ms as a function of (scaled) distance z, in the non-interacting case. The central condensate centred at $z = 0$ has been subtracted away. Different curves correspond to increasing well depths, from shallow (intermediate thickness) to large (thick curve). Redrawn from Chiofalo *et al.* [6].

been subtracted and the curves are shown for three values of the lattice-barrier height. The latter is determined in the experiment by the intensity of the optical laser beams.

When a monochromatic oscillating drive is added to the constant drive, one finds that the equations governing this type of coherent BEC transport in the linear non-dissipative regime can be mapped into those for the a.c. superconducting current flowing across a weak-link Josephson junction [17]. In essence, the voltage drop across the junction is replaced by the product of the constant component of the force times the lattice spacing, and the frequency of Bloch oscillations is in resonance with integer multiples of the oscillating drive frequency. The relevant experiment has been carried out by Burger *et al.* [18] on a BEC confined in a magnetic trap plus an optical lattice, the oscillating force being very simply generated by a rapid shift of the centre of the magnetic trap.

5. RECENT STUDIES OF COLLECTIVE EXCITATIONS IN TRAPPED GASEOUS SUPERFLUIDS

We turn next to present summaries of two very recent theoretical studies of collective modes in confined superfluids, but now using more general many-body theory. The first of these studies relates to a Bose gas of ^{87}Rb atoms at different temperatures, where experimental frequencies and damping rates of such collective excitations are available. The second study is addressed to the behaviour of collective-mode frequencies in the BEC–BCS crossover that we have referred to in Section 3 for Fermi gases with attractive interactions leading to pairing of fermions with opposite spins.

5.1. Collective Modes in ^{87}Rb Gas at Various Temperatures

Soon after the achievement of Bose–Einstein condensation in trapped atomic gases, it proved possible to carry out very accurate measurements of the frequencies and damping rates of collective excitations in response to suitable external modulations of the trap (see Onofrio *et al.* [19] and references given there). Mean-field theory fails when faced with the data for the temperature dependence of these quantities, as reported by the group at JILA [20] from measurements on ^{87}Rb gas at various temperatures. The key issue in investigations transcending mean-field approaches is to give a full dynamical description of both the condensate and the thermal cloud and of their mutual interactions [21]. While the dynamics of the Bose-condensed atoms can be well reproduced by a single non-linear GP equation, treating the evolution of the non-condensate is a much more delicate problem.

The study of Liu *et al.* [7] treats the motions of both condensate and non-condensate on an equal footing at the level of a generalized random-phase approximation (RPA), by also including the dynamical coupling between fluctuations in the thermal cloud. Their work reduces in the appropriate limit to the earlier approaches based on the second-order Beliaev-Popov approximation [22–24], which fail to account for the JILA observations. The crucial step forward lies in allowing for the anomalous density fluctuations, which turn out to be essential in determining the temperature dependence of the monopolar modes of the trapped gas. These calculations also satisfy the generalized Kohn theorem for the dipolar excitations accurately. Mean-field theory serves however to yield quantitative accuracy for the quadrupolar modes. The simplifying assumption of isotropic confinement made by Liu *et al.* will need transcending for a fully quantitative contact with the data from the JILA experiments.

5.2. Collective Modes of a Fermion Superfluid in the BCS–BEC Crossover

Current experimental studies on ultracold Fermi gases are now rapidly advancing towards the achievement of superfluid states, and Bose–Einstein condensation of spin-paired dimers has already been realized (see [25,26] and earlier references there). In the use of atomic gases, one can exploit a Feshbach resonance to change the magnitude and the sign of the coupling strength. As one crosses such a resonance, the *s*-wave scattering length a goes from large positive to negative values, and thereby it becomes feasible to explore the crossover from the BEC of bound fermion pairs to the BCS state of Cooper pairs. This issue was first raised theoretically in the early study of Nozières and Schmitt-Rink [27] (see also Leggett [28]).

The mean-field theory for the crossover extends the usual BCS gap equation

$$\frac{4\pi\hbar^2 a}{m}\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2\varepsilon_k}-\frac{1}{2E_k}\right]=1 \tag{14}$$

and number equation

$$\int\frac{d\mathbf{k}}{(2\pi)^3}\left(1-\frac{\varepsilon_k-\mu}{E_k}\right)=n \tag{15}$$

to the whole interaction region. Here $\varepsilon_k=\hbar^2k^2/2m$, $E_k=\sqrt{(\varepsilon_k-\mu)^2+\Delta^2}$, and Δ is the gap parameter. These equations are to be solved for μ and Δ with a given choice of the dimensionless coupling parameter $k_Fa=(3\pi^2n)^{1/3}a$. In the weak-coupling limit ($k_Fa\to 0^-$) they give back the standard BCS result, and when ($k_Fa\to 0^+$) they correctly reproduce the binding energy of fermion pairs at leading order [29]. In the unitary limit ($a\to\pm\infty$) the model gives $\mu(n)\propto n^{2/3}$, scaling as predicted by the universality hypothesis [30].

Relating to such a crossover, Hu *et al.* [8] in a very recent theoretical study have evaluated the frequencies of the collective breathing modes and the anisotropic expansion rate of a gaseous Fermi superfluid prepared in an axially symmetric harmonic trap, at varying coupling strength for opposite-spin fermions across a Feshbach resonance. The dynamics of the gaseous superfluid is treated by hydrodynamic equations involving a microscopic mean-field expression for the compressibility, which is calculated from the equilibrium density profile. A scaling Ansatz and a local-density approximation are used to obtain simple analytical expressions for the mode frequencies in the experimentally relevant case of a highly elongated cigar-shaped trap and in other cases of trap shape. These theoretical results demonstrate non-monotonic behaviour of the mode frequencies in the crossover region and are in very good quantitative agreement with current measurements of the transverse breathing mode by Kinast *et al.* [31] and of the axial breathing mode by Grimm *et al.* [32].

6. SUMMARY AND SOME PROPOSED FUTURE DIRECTIONS

Having set down the energy variational principle which leads by minimization to the static GP differential equation, with one application of this equation to a vortex array, the following section summarizes a recently proposed integral equation which should transcend Eq. (2) for the case of composite bosons formed from strongly coupled fermions. The starting point for its derivation is taken as the Bogoliubov–de Gennes equations for superfluid fermions, and it remains a matter for future work to ascertain the role of the non-locality effects that have thereby been introduced into the theory. The theoretical framework still is based on the two-body scattering problem in a dilute quantum gas, which has recently been critically discussed by Leggett in relation to a genuine many-body treatment of a Bose superfluid with repulsive interactions.

Sections 4 and 5 are then concerned with the dynamics of inhomogeneous superfluids. While Section 4 is addressed to conventional time-dependent GP theory, along with a numerical study of BEC drop emission under gravity from an optical lattice, Section 5 is concerned with applications of more advanced theories to collective excitations of superfluid gases in two different contexts. First of all, collective modes in a boson gas at different temperatures below the critical temperature for Bose–Einstein condensation are a focus, since frequencies and damping rates have been measured and cannot be understood with a mean-field framework. The motions of both condensate and non-condensate need to be treated on an equal footing, as has been done in a generalized-RPA framework demonstrating a crucial role of anomalous density fluctuations in the coupling between the two components of a thermally excited Bose gas. Then the second issue concerns the behaviour of the mode frequencies in a harmonically trapped Fermi superfluid as the coupling strength for fermion pairing is varied from the BCS weak-coupling limit to the situation in which a Bose–Einstein condensate is generated from composite bosons formed from bound fermion pairs. Here a mean-field approach has been used to treat the dynamics of the fluid as a function of the fermion–fermion coupling strength across a Feshbach resonance, and has been shown to account quantitatively for the emerging experimental data in the crossover region.

Following this summary, we shall conclude this review by referring to a few future directions which appear attractive at the time of writing.

6.1. Some Areas Promising Further Fruitful Interaction with Experiment

Intimately connected with the GP equations is the recent study of Bao *et al.* [33] on three-dimensional simulation of jet formation in collapsing condensates. In particular, these authors report a fully three-dimensional numerical solution of the GP equation, including three-body losses, to describe the behaviour of collapsing and exploding condensates with system parameters relevant to the experiments of Donley *et al.* [34]. Bao *et al.* determine the three-body loss rate from the number of remaining condensate atoms and collapse times, and thereby obtain only one possible value that agrees with the experimental results. They then study the formation of jet atoms by interrupting the collapse: good agreement with experiment results.

Arnold *et al.* [35] have been concerned with diffraction-limited focussing of Bose–Einstein condensates. They achieve adjustable magnetic reflection and focussing of a BEC of ^{87}Rb atoms and discuss a simple Thomas–Fermi model and a Monte Carlo method for analyzing the bouncing. Both methods turn out to be in close accord with the observed condensate evolution. In addition, the theory predicts very tight condensate focussing, and atomic matter-wave diffraction ought to be accessible experimentally.

Arnold *et al.* [35] also comment that when one begins to probe extremely small length scales, both simple theories that they currently employ will cease to be valid. Such experiments for the future, they conclude, will need to be compared with the predictions of the three-dimensional time-dependent GP equation. However, they also conclude that the simpler Thomas–Fermi model should be adequate if a sufficiently harmonic magnetic mirror is used, and that the Monte Carlo approach may be adequate for describing condensate behaviour prior to and immediately after focussing provided the tightness of focus does not induce atomic matter-wave diffraction.

Also, we wish to mention here that the dynamics of bright matter-wave solitons (see also [1]) in a BEC with inhomogeneous scattering length has been the subject of a recent study by Abdullaev *et al.* [36]. A rich dynamics in the interaction between the soliton and an inhomogeneity is observed. By numerical simulation of one- and three-dimensional GP equations, these workers study trapping, reflection, and transmission of the soliton due to an impurity. Conditions are also given for the collapse of the bright solitary wave, considering a quasi-one-dimensional BEC with attractive local inhomogeneity. Another area of potential application here is that of non-linear photonic crystals [37].

As the final remark, current activity in this field of the statics and dynamics of inhomogeneous superfluids is huge and makes it abundantly clear that further many-body theory in this area will be of importance. This, we believe, will be especially so if its interactions with exciting new fields of experimental study now coming to full fruition are made a major focus for the future.

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